Life after Gabis

[Columbia UMS
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Based in part on material of
Leo Goldmakher, Leo C. Stein

We start with the famous Abel-Ruffini theorem.

Thm (Ruffini 1799, Abel 1824)

There is no formula for the roots of a general 5^{th} degree polynomial involving only arithmetic operations $(+, -, \cdot, \div)$ and radicals $\sqrt[n]{\ }$.

Today: - See a modern proof due to Arnol'd that uses topology in place of Galois thy.

- See how complex dynamics can show that even approximate solutions to polynomials are inherently constrained.

The spaces $\mathrm{Poly}_n\mathbb{C}$, $\mathrm{Root}_n\mathbb{C}$

Two natural parametrizations of n^{th} degree monic polynomials$/\mathbb{C}$:

$$P(z) = z^n + a_{n-1}z + \cdots + a_1 z + a_0$$

① $\mathrm{Poly}_n\mathbb{C}$: Parametrize the coefficients.

$$(a_0, \ldots, a_{n-1}) \longmapsto z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

② $\mathrm{Root}_n\mathbb{C}$: Parametrize the roots:

$$(r_1, \ldots, r_n) \longmapsto (z-r_1)\cdots(z-r_n)$$

Note both $\mathrm{Poly}_n\mathbb{C}$, $\mathrm{Root}_n\mathbb{C} \cong \mathbb{C}^n$.

Recall that you can express the coefficients as symmetric functions of the roots, eg.

$$a_{n-1} = -\sum r_i, \quad a_0 = (-1)^n r_1 \cdots r_n.$$

$$a_{n-2} = \sum r_i r_j \quad i \neq j$$

So there is a map

$$\Psi : \mathrm{Root}_n \mathbb{C} \longrightarrow \mathrm{Poly}_n \mathbb{C}.$$

If $r_1, \ldots, r_n$ are all distinct, then Ψ has degree $n!$

(The $n!$ Points $(r_{\sigma(1)}, \ldots, r_{\sigma(n)})$ for $\sigma \in S_n$ are distinct in $\mathrm{Root}_n \mathbb{C}$, but map to same $P \in \mathrm{Poly}_n \mathbb{C}$).

Abstractly, root finding is about finding a map the other way: $\mathrm{Poly}_n \mathbb{C} \to \mathrm{Root}_n \mathbb{C}$.

Arnol'd's Argument

Warmup level 1 : $\sqrt{z}$

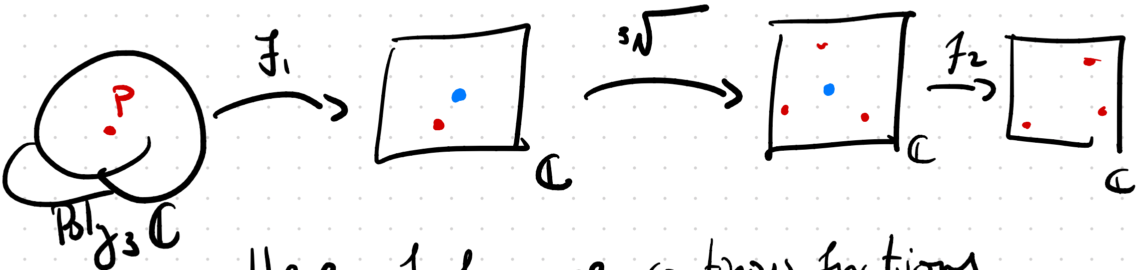
Look at applet for $n=2$.
We see there are <u>loops</u> in $\mathrm{Poly}_2\mathbb{C}$ that do not lift to <u>loops</u> in $\mathrm{Roots}_2\mathbb{C}$. (only <u>paths</u>).

This tells us something about solving quadratics:

There is no formula involving continuous single-valued functions that can pick out <u>one</u> root of a quadratic.

Warmup level 2: For cubics, a single radical won't do!

Suppose we had a cubic formula that involved only one radical:



Here f_1, f_2 are continous fuctions.

Let's examine what happens as we move p along a loop.

Applet: Can easily realize (123) as a permutation.

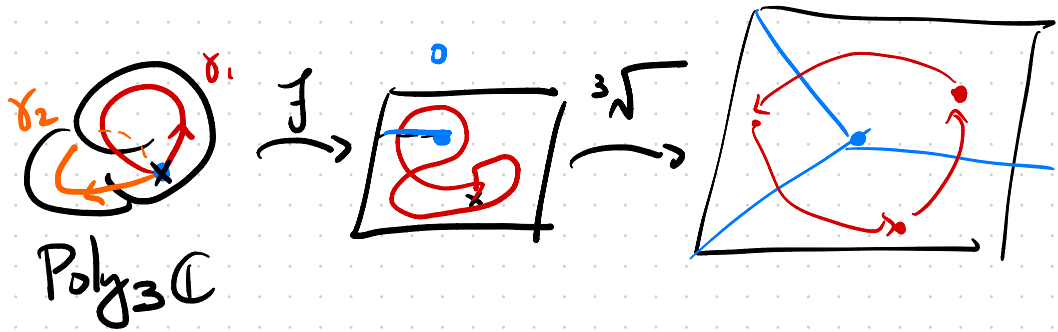
By "reverse-engineering", can make (12) also.

In fact, given <u>any</u> permutation σ of the roots, can build a loop γ_σ in $\mathrm{Poly}_n\,\mathbb{C}$ that realizes it!

Commutators: Given paths γ_1, γ_2, the commutator is the concatenation

$\gamma_1 \gamma_2 \overline{\gamma_1} \overline{\gamma_2}$, where $\overline{\gamma}$ is γ run backwards.

Let's look at how commutators would behave under radicals.

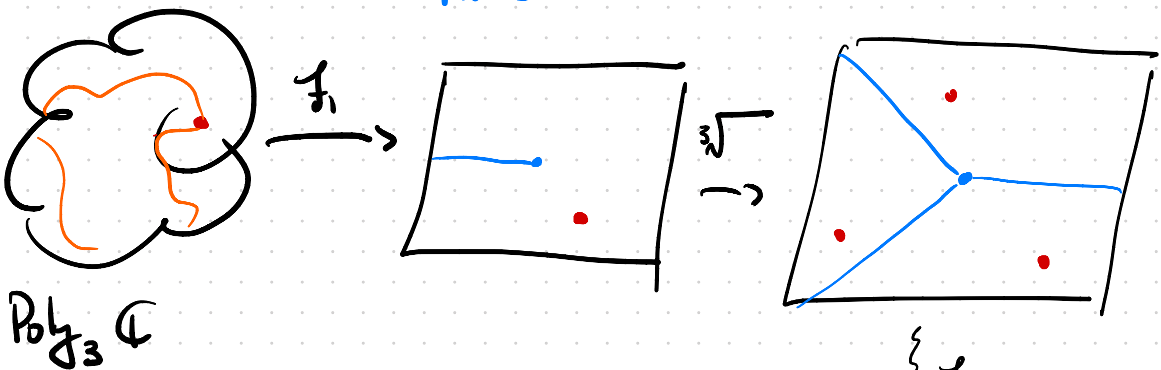


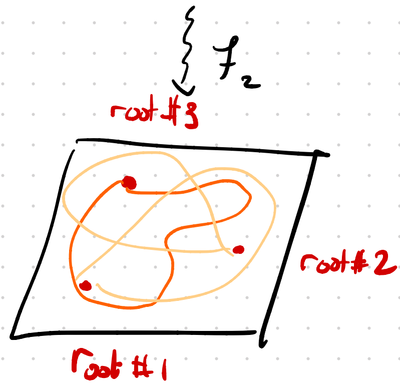
See in applet: Permutation is $(123)^k$, where k counts the winding number of γ around 0.

But observe that any commutator has WN $= 0$.

Picture:

Suppose we had a cubic formula
with one radical:





Then as we moved p around
<u>any</u> commutator-loop,
we'd see the roots complete
a <u>loop</u>, not a path:

<u>But</u>, notice $[\gamma_{123}, \gamma_{12}]$ induces $[(123), (12)]$
$= (132)$,

So if you do this commutator-loop,
you fail to lift!

Quintics: Just an elaboration of same ideas.

Notice $[(12345),(135)] = (241)(153)$

$= (15324)$,

5-cycles are commutators!

$\Rightarrow$ Any quintic formula in radicals would need <u>nested radicals</u>.

Recall: 3-cycles are commutators, too.

So can write (12345) as $[[\gamma_1,\gamma_2],[\gamma_3,\gamma_4]]$,

i.e. a double commutator.

$\Rightarrow$ Any quintic formula in radicals would need <u>three layers of nesting</u>.

(Each of $[\gamma_1,\gamma_2]$, $[\gamma_3,\gamma_4]$ <u>themselves</u> lift as <u>loops</u> under the first $\sqrt[n]{\cdot}$,

So $[[\gamma_1,\gamma_2],[\gamma_3,\gamma_4]]$ survives a double radical.)

Endgame:

We wrote 5-cycle = [5-cycle, 3-cycle].

See: $[(123),(145)] = (245)(154) = (124)$

So: 3-cycle = [3-cycle, 3-cycle].

So now you can repeat ad inf.:

A 5-cycle can be expressed as an n-fold iterated commutator for any n!

$\Rightarrow$ Any quintic formula requires $>n$ nested radicals, for all n!

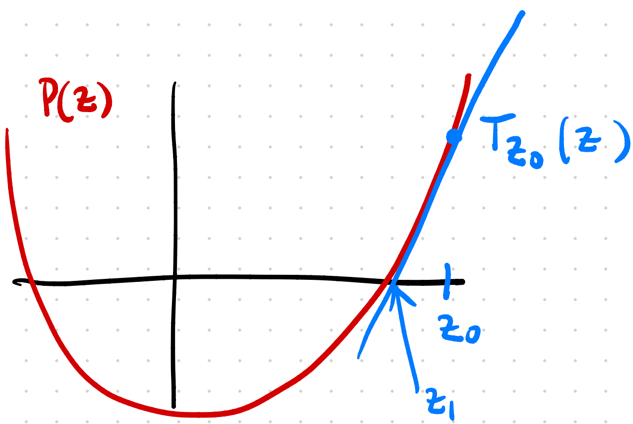
$\Rightarrow$ No quintic formula in radicals!

Part 2: Approximations

In practice, often an approximate solution to $p(z)$ is good enough.

Recall Newton's method:

Instead of solving $p(z)$, solve $T_{z_0}(z)$, the tangent line to p at $z=z_0$, then iterate.



$$z_1 = z_0 - \frac{p(z_0)}{p'(z_0)}$$

Newton's method is a purely-iterative algorithm: (PIA)

Given $P(z)$, obtain a self-map of $\hat{\mathbb{C}}$:

$$R_p(z) = z - \frac{P(z)}{P'(z)}$$

A PIA is generally-convergent if for a full-measure set of initial guesses z_0, the sequence $\{z_0, R_p(z_0), R_p^2(z_0), \ldots\} \to \lambda$ a root of P.

Exercise/Fact: Newton's method is generally-convergent for quadratics.

What about cubics?

In applet: Examine $P_\lambda(z) = (z-\lambda)(z+\lambda)(z-1)$

for $\lambda = 0.589 + .605i$.

See initial guesses

$z_0 = 0 \to 1$

$z_0 = 0.1 \to 1$

$z_0 = 0.2 \to 1$

$z_0 = 0.25 \to 1$ Slowly!

$z_0 = 0.26 \to -\lambda$!

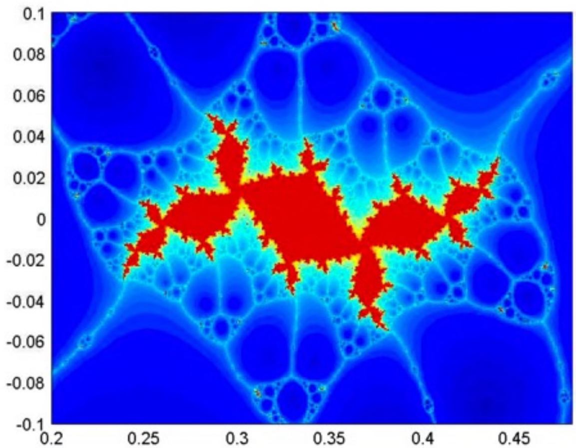
$z_0 = 0.27 \to ???$!!

$z_0 = 0.3 \to \lambda$

$z_0 = 0.32 \to ???$

What's going on here?

Red zone: Zone of chaos!



(Credit: Shannon N. Miller)

Very plausible: red set has positive measure.

⟶ Newton's method fails: not generally-convergent!

Enter McMullen:

Thm (McMullen, '87)

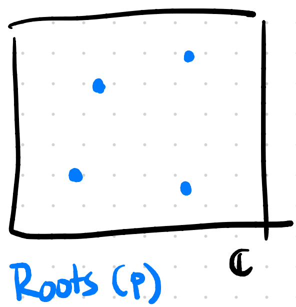
For polynomials of degree $d \geq 4$, no generally-conv. PIA algorithm whatsoever exists!

In followup work, he gave a more explicit obstruction.

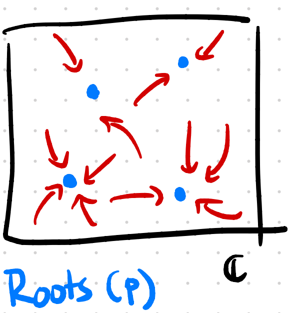
We'll close with a discussion of this, since it is very much in the spirit of Arnol'd's proof from part 1.

"Braiding the attractor"

Consider a polynomial $p(z)$:



Say we had a PIA $R_p(z)$ that worked:



Roots(p): the attractor of R_p.

Now imagine we had one that worked
in general.

We could take a loop P_λ of polynomials.

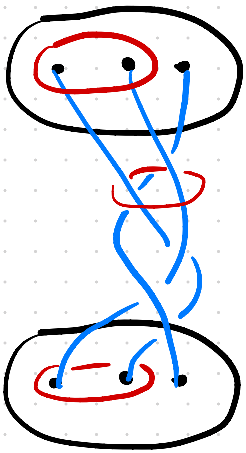
This would give us a loop of rational maps
with a loop of attractors.

Notice that this forms a braid.

Recall from part 1: Given any braid, we
can realize it as the loop of roots of
polynomials.

McMullen shows that there are strong constraints on the kinds of braids that can arise as loops of attractors!

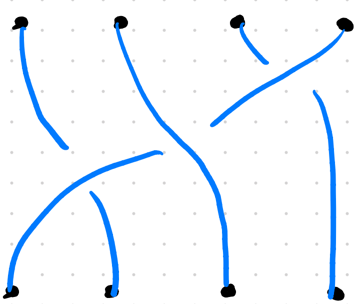
The constraint is <u>roughly</u> this:



There must be some curve left invariant by (some power of) the braid.

(For technical reasons, need ≥4 strands, too).

But there are braids with no such fixed loops:



(Exercise!)

So if your roots trace out this braid, you can't have a DIA along it!